Vektoranalysis

Teil V

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I n h a l t

Anhang

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In diesem Anhang werden zunächst die Beweise der elementaren Rechengesetze für die Differentialoperatoren Gradient, Divergenz und Rotation aufgeführt. Dabei gehe ich auch auf die Anwendung des Differentialoperators Nabla ein. Danach beweise ich die Rechengesetze für die Kombinationen der Differentialoperatoren. Dabei werden die Rechengesetze der Analysis (skalarer Funktionen) als bekannt vorausgesetzt.

Zur deutlichen und auffälligen Unterscheidung verwende ich dabei für skalare Ortsfunktionen die Buchstaben $f = f(x, y, z)$, $g = g(x, y, z)$ usw., für vektorielle Ortsfunktionen die Bezeichnungen $\boldsymbol{v} = \boldsymbol{v}(x, y, z)$, $\boldsymbol{w} = \boldsymbol{w}(x, y, z)$ usw.

1 Elementare Rechengesetze für die Differentialoperatoren

1.1 Gradient

$$\begin{aligned}
\operatorname{grad}(f+g) &\equiv \frac{\partial(f+g)}{\partial x}\boldsymbol{e}_1 + \frac{\partial(f+g)}{\partial y}\boldsymbol{e}_2 + \frac{\partial(f+g)}{\partial z}\boldsymbol{e}_3 \\
&= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x}\right)\boldsymbol{e}_1 + \left(\frac{\partial f}{\partial y} + \frac{\partial g}{\partial y}\right)\boldsymbol{e}_2 + \left(\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z}\right)\boldsymbol{e}_3 \\
&= \frac{\partial f}{\partial x}\boldsymbol{e}_1 + \frac{\partial f}{\partial y}\boldsymbol{e}_2 + \frac{\partial f}{\partial z}\boldsymbol{e}_3 + \frac{\partial g}{\partial x}\boldsymbol{e}_1 + \frac{\partial g}{\partial y}\boldsymbol{e}_2 + \frac{\partial g}{\partial z}\boldsymbol{e}_3 \\
&= \operatorname{grad} f + \operatorname{grad} g,
\end{aligned}$$

oder

$$\nabla(f+g) = \nabla f + \nabla g.$$

$$\begin{aligned}
\operatorname{grad}(f \cdot g) &\equiv \frac{\partial(f \cdot g)}{\partial x}\boldsymbol{e}_1 + \frac{\partial(f \cdot g)}{\partial y}\boldsymbol{e}_2 + \frac{\partial(f \cdot g)}{\partial z}\boldsymbol{e}_3 \\
&= \left(\frac{\partial f}{\partial x}g + f\frac{\partial g}{\partial x}\right)\boldsymbol{e}_1 + \left(\frac{\partial f}{\partial y}g + f\frac{\partial g}{\partial y}\right)\boldsymbol{e}_2 + \left(\frac{\partial f}{\partial z}g + f\frac{\partial g}{\partial z}\right)\boldsymbol{e}_3 \\
&= \left(\frac{\partial f}{\partial x}\boldsymbol{e}_1 + \frac{\partial f}{\partial y}\boldsymbol{e}_2 + \frac{\partial f}{\partial z}\boldsymbol{e}_3\right)g + f\left(\frac{\partial g}{\partial x}\boldsymbol{e}_1 + \frac{\partial g}{\partial y}\boldsymbol{e}_2 + \frac{\partial g}{\partial z}\boldsymbol{e}_3\right) \\
&= (\operatorname{grad} f)g + f\operatorname{grad} g,
\end{aligned}$$

oder

$$\nabla(f \cdot g) = (\nabla f)g + f\nabla g.$$

$$\begin{aligned}
\operatorname{grad} f^n &\equiv \frac{\partial f^n}{\partial x}\boldsymbol{e}_1 + \frac{\partial f^n}{\partial y}\boldsymbol{e}_2 + \frac{\partial f^n}{\partial z}\boldsymbol{e}_3 \\
&= n\, f^{n-1}\frac{\partial f}{\partial x}\boldsymbol{e}_1 + n\, f^{n-1}\frac{\partial f}{\partial y}\boldsymbol{e}_2 + n\, f^{n-1}\frac{\partial f}{\partial z}\boldsymbol{e}_3 \\
&= n\, f^{n-1}\left(\frac{\partial f}{\partial x}\boldsymbol{e}_1 + \frac{\partial f}{\partial y}\boldsymbol{e}_2 + \frac{\partial f}{\partial z}\boldsymbol{e}_3\right) \\
&= n\, f^{n-1}\operatorname{grad} f^n,
\end{aligned}$$

oder

$$\nabla f^n = n\, f^{n-1}\nabla f^n.$$

1.2 Divergenz

$$\begin{aligned}
\operatorname{div}(\boldsymbol{v}+\boldsymbol{w}) &\equiv \frac{\partial(\boldsymbol{v}+\boldsymbol{w})_x}{\partial x} + \frac{\partial(\boldsymbol{v}+\boldsymbol{w})_y}{\partial y} + \frac{\partial(\boldsymbol{v}+\boldsymbol{w})_z}{\partial z} \\
&= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} + \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z} \\
&= \operatorname{div}\boldsymbol{v} + \operatorname{div}\boldsymbol{w}
\end{aligned}$$

oder

$$\nabla\cdot(\boldsymbol{v}+\boldsymbol{w}) = \nabla\cdot\boldsymbol{v} + \nabla\cdot\boldsymbol{w}.$$

$$\begin{aligned}
\operatorname{div}(f\,\boldsymbol{v}) &\equiv \frac{\partial(f\,\boldsymbol{v})_x}{\partial x} + \frac{\partial(f\,\boldsymbol{v})_y}{\partial y} + \frac{\partial(f\,\boldsymbol{v})_z}{\partial z} = \frac{\partial(f v_x)}{\partial x} + \frac{\partial(f v_y)}{\partial y} + \frac{\partial(f v_z)}{\partial z} \\
&= \frac{\partial f}{\partial x}v_x + f\frac{\partial v_x}{\partial x} + \frac{\partial f}{\partial y}v_y + f\frac{\partial v_y}{\partial y} + \frac{\partial f}{\partial z}v_z + f\frac{\partial v_z}{\partial z} \\
&= \left(\frac{\partial f}{\partial x}\boldsymbol{e}_1 + \frac{\partial f}{\partial y}\boldsymbol{e}_2 + \frac{\partial f}{\partial z}\boldsymbol{e}_3\right)\cdot(v_x\boldsymbol{e}_1 + v_y\boldsymbol{e}_2 + v_z\boldsymbol{e}_3) + f\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) \\
&= (\operatorname{grad} f)\cdot\boldsymbol{v} + f\operatorname{div}\boldsymbol{v}
\end{aligned}$$

oder

$$\nabla\cdot(f\,\boldsymbol{v}) = \nabla f\cdot\boldsymbol{v} + f(\nabla\cdot\boldsymbol{v}).$$

$$
\begin{aligned}
\operatorname{div}(\boldsymbol{v}\times\boldsymbol{w}) &\equiv \frac{\partial(\boldsymbol{v}\times\boldsymbol{w})_x}{\partial x}+\frac{\partial(\boldsymbol{v}\times\boldsymbol{w})_y}{\partial y}+\frac{\partial(\boldsymbol{v}\times\boldsymbol{w})_z}{\partial z}\\
&=\frac{\partial}{\partial x}(v_y w_z - v_z w_y)+\frac{\partial}{\partial y}(v_z w_x - v_x w_z)+\frac{\partial}{\partial z}(v_x w_y - v_y w_x)\\
&=\frac{\partial v_y}{\partial x}w_z + v_y\frac{\partial w_z}{\partial x}-\frac{\partial v_z}{\partial x}w_y - v_z\frac{\partial w_y}{\partial x}\\
&\quad+\frac{\partial v_z}{\partial y}w_x + v_z\frac{\partial w_x}{\partial y}-\frac{\partial v_x}{\partial y}w_z - v_x\frac{\partial w_z}{\partial y}\\
&\quad+\frac{\partial v_x}{\partial z}w_y + v_x\frac{\partial w_y}{\partial z}-\frac{\partial v_x}{\partial z}w_y - v_y\frac{\partial w_x}{\partial z}\\
&=\left[\left(\frac{\partial v_z}{\partial y}-\frac{\partial v_y}{\partial z}\right)\boldsymbol{e}_1+\left(\frac{\partial v_x}{\partial z}-\frac{\partial v_z}{\partial x}\right)\boldsymbol{e}_2+\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\boldsymbol{e}_3\right]\cdot(w_x\boldsymbol{e}_1+w_y\boldsymbol{e}_2+w_z\boldsymbol{e}_3)\\
&\quad-(v_x\boldsymbol{e}_1+v_y\boldsymbol{e}_2+v_z\boldsymbol{e}_3)\left[\left(\frac{\partial w_z}{\partial y}-\frac{\partial w_y}{\partial z}\right)\boldsymbol{e}_1+\left(\frac{\partial w_x}{\partial z}-\frac{\partial w_z}{\partial x}\right)\boldsymbol{e}_2+\left(\frac{\partial w_y}{\partial x}-\frac{\partial w_x}{\partial y}\right)\boldsymbol{e}_3\right]\\
&=(\operatorname{rot}\boldsymbol{v})\cdot\boldsymbol{w}-\boldsymbol{v}\cdot\operatorname{rot}\boldsymbol{w}
\end{aligned}
$$

oder

$$\nabla\cdot(\boldsymbol{v}\times\boldsymbol{w})=(\nabla\times\boldsymbol{v})\cdot\boldsymbol{w}-\boldsymbol{v}\cdot(\nabla\times\boldsymbol{w}).$$

1.3 Rotation

$$
\begin{aligned}
\operatorname{rot}(\boldsymbol{v}+\boldsymbol{w}) &\equiv \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ (\boldsymbol{v}+\boldsymbol{w})_x & (\boldsymbol{v}+\boldsymbol{w})_y & (\boldsymbol{v}+\boldsymbol{w})_z \end{vmatrix} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ v_x & v_y & v_z \end{vmatrix} + \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ w_x & w_y & w_z \end{vmatrix}\\
&=\operatorname{rot}\boldsymbol{v}+\operatorname{rot}\boldsymbol{w}
\end{aligned}
$$

oder

$$\nabla\times(\boldsymbol{v}+\boldsymbol{w})=\nabla\times\boldsymbol{v}+\nabla\times\boldsymbol{w}.$$

$$\operatorname{rot}(f\,\boldsymbol{v}) \equiv \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ f\,v_x & f\,v_y & f\,v_z \end{vmatrix}$$

$$= \left(\frac{\partial(f\,v_z)}{\partial y} - \frac{\partial(f\,v_y)}{\partial z}\right)\boldsymbol{e}_1 + \left(\frac{\partial(f\,v_x)}{\partial z} - \frac{\partial(f\,v_z)}{\partial x}\right)\boldsymbol{e}_2 + \left(\frac{\partial(f\,v_y)}{\partial x} - \frac{\partial(f\,v_x)}{\partial y}\right)\boldsymbol{e}_3$$

$$= \left(\frac{\partial f}{\partial y}v_z + f\,\frac{\partial v_z}{\partial y} - \frac{\partial f}{\partial z}v_y - f\,\frac{\partial v_y}{\partial z}\right)\boldsymbol{e}_1$$

$$+ \left(\frac{\partial f}{\partial z}v_x + f\,\frac{\partial v_x}{\partial z} - \frac{\partial f}{\partial x}v_z - f\,\frac{\partial v_z}{\partial x}\right)\boldsymbol{e}_2$$

$$+ \left(\frac{\partial f}{\partial x}v_y + f\,\frac{\partial v_y}{\partial x} - \frac{\partial f}{\partial y}v_x - f\,\frac{\partial v_x}{\partial y}\right)\boldsymbol{e}_3$$

$$= \left(\frac{\partial f}{\partial y}v_z - \frac{\partial f}{\partial z}v_y\right)\boldsymbol{e}_1 + \left(\frac{\partial f}{\partial z}v_x - \frac{\partial f}{\partial x}v_z\right)\boldsymbol{e}_2 + \left(\frac{\partial f}{\partial x}v_y - \frac{\partial f}{\partial y}v_x\right)\boldsymbol{e}_3$$

$$+ f\left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\boldsymbol{e}_1 + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\boldsymbol{e}_2 + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\boldsymbol{e}_3\right]$$

$$= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} + f\,\operatorname{rot}\boldsymbol{v}$$

$$= (\operatorname{grad} f)\times\boldsymbol{v} + f\,\operatorname{rot}\boldsymbol{v}$$

oder

$$\nabla\times(f\,\boldsymbol{v}) = \nabla f\times\boldsymbol{v} + f\,(\nabla\times\boldsymbol{v}).$$

2 Zweifache Differentialoperationen

Durch Kombination von zwei Differentialoperatoren (Gradient, Divergenz, Rotation) werden die zweiten partiellen Ableitungen der betreffenden Funktion(en) gebildet. Wegen der Natur (Skalar oder Vektor) und wegen der Anwendbarkeit (auf skalare Funktionen oder auf Vektorfunktionen) sind nur bestimmte Kombinationen möglich. Diese sind:

$$
\begin{aligned}
\operatorname{grad}\,\operatorname{div}\boldsymbol{v} &= \nabla(\nabla\cdot\boldsymbol{v}),\\
\operatorname{div}\operatorname{grad}\,f &= \nabla\cdot(\nabla f),\\
\operatorname{div}\operatorname{rot}\boldsymbol{v} &= \nabla\cdot(\nabla\times\boldsymbol{v}),\\
\operatorname{rot}\operatorname{grad}\,f &= \nabla\times\operatorname{grad}\,f,\\
\operatorname{rot}\operatorname{rot}\boldsymbol{v} &= \nabla\times(\nabla\times\boldsymbol{v}).
\end{aligned}
$$

Wir untersuchen nun diese Kombinationen:

$$
\begin{aligned}
\operatorname{grad}\,\operatorname{div}\boldsymbol{v} \equiv \nabla(\nabla\cdot\boldsymbol{v}) &\equiv \frac{\partial}{\partial x}\operatorname{div}\boldsymbol{v}\,\boldsymbol{e}_1 + \frac{\partial}{\partial y}\operatorname{div}\boldsymbol{v}\,\boldsymbol{e}_2 + \frac{\partial}{\partial z}\operatorname{div}\boldsymbol{v}\,\boldsymbol{e}_3\\
&= \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial y\,\partial x} + \frac{\partial^2 v_z}{\partial z\,\partial x}\right)\boldsymbol{e}_1 + \left(\frac{\partial^2 v_x}{\partial x\partial y} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_z}{\partial z\,\partial x}\right)\boldsymbol{e}_2\\
&\quad + \left(\frac{\partial^2 v_x}{\partial x\partial z} + \frac{\partial^2 v_y}{\partial y\,\partial z} + \frac{\partial^2 v_z}{\partial z^2}\right)\boldsymbol{e}_3.
\end{aligned}
$$

Weitere Vereinfachungen oder symbolische Abkürzungen sind nicht möglich.

$$
\operatorname{div}\operatorname{grad}\,f \equiv \nabla\cdot\nabla f \equiv \operatorname{div}\left(\frac{\partial f}{\partial x}\boldsymbol{e}_1 + \frac{\partial f}{\partial y}\boldsymbol{e}_2 + \frac{\partial f}{\partial z}\boldsymbol{e}_3\right) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \Delta f
$$

Δ: LAPLACE-Operator

Für die Anwendung auf eine skalare Ortsfunktion gilt also $\nabla\cdot\nabla = \Delta$.

$$
\begin{aligned}
\operatorname{div}\operatorname{rot}\boldsymbol{v} \equiv \nabla\cdot(\nabla\times\boldsymbol{v}) &\equiv \operatorname{div}\left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\boldsymbol{e}_1 + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\boldsymbol{e}_2 + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\boldsymbol{e}_3\right]\\
&= \frac{\partial}{\partial x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) = 0.
\end{aligned}
$$

Beweis durch Ausrechnen und Beachtung des Satzes von SCHWARZ, der besagt, dass bei Stetigkeit der zweiten Ableitungen die Reihenfolge der Differentiationen beliebig ist.

Physikalisch besagt der Satz, dass jedes Rotationsfeld quellenfrei ist.

Man sieht, dass in diesem Fall der symbolische Vektor Nabla der Regel folgt, dass das Skalarprodukt zweier aufeinander senkrechter Vektoren null ist.

$$\operatorname{rot}\operatorname{grad} f \equiv \nabla\times(\nabla f) \equiv \operatorname{rot}\left(\frac{\partial f}{\partial x}\boldsymbol{e}_1 + \frac{\partial f}{\partial y}\boldsymbol{e}_2 + \frac{\partial f}{\partial z}\boldsymbol{e}_3\right) = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 f}{\partial z \partial y} - \frac{\partial^2 f}{\partial y \partial z}\right)\boldsymbol{e}_1 + \left(\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right)\boldsymbol{e}_2 + \left(\frac{\partial^2 f}{\partial y \partial x} - \frac{\partial^2 f}{\partial x \partial y}\right)\boldsymbol{e}_3 = 0.$$

Die Begründung ist dieselbe wie oben (Satz von SCHWARZ).
Physikalisch besagt der Satz, dass jedes Gradientenfeld rotationsfrei (wirbelfrei) ist.

$$\operatorname{rot}\operatorname{rot}\boldsymbol{v} \equiv \nabla\times(\nabla\times\boldsymbol{v}) \equiv \operatorname{rot}\begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \operatorname{rot}\left[\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\boldsymbol{e}_1 + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\boldsymbol{e}_2 + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\boldsymbol{e}_3\right]$$

$$= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial v_z}{\partial y} - \dfrac{\partial v_y}{\partial z} & \dfrac{\partial v_x}{\partial z} - \dfrac{\partial v_z}{\partial x} & \dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y} \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\right]\boldsymbol{e}_1$$

$$+ \left[\frac{\partial}{\partial z}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\right]\boldsymbol{e}_2$$

$$+ \left[\frac{\partial}{\partial x}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\right]\boldsymbol{e}_3$$

$$
= \left(\frac{\partial^2 v_y}{\partial x \partial y} - \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial^2 v_z}{\partial x \partial z} \right) \boldsymbol{e}_1 + \left(\frac{\partial^2 v_z}{\partial y \partial z} - \frac{\partial^2 v_y}{\partial z^2} - \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_x}{\partial y \partial x} \right) \boldsymbol{e}_2
$$

$$
+ \left(\frac{\partial^2 v_x}{\partial z \partial x} - \frac{\partial^2 v_z}{\partial x^2} - \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_y}{\partial z \partial y} \right) \boldsymbol{e}_3
$$

Dieser Vektor kann so umgeformt werden, dass daraus ein Ausdruck wird, der sich später in der Elektrodynamik als sehr nützlich erweist: Zu seinen Komponenten werden jeweils drei Terme *A*, *B*, *C* addiert, die am Ende wieder subtrahiert werden, sodass daraus die Differenz zweier Vektoren wird. Der erste davon ist der Vektor grad div ***v***. Der letzte Term ist der Vektor, der durch Anwendung des LAPLACE-Operators auf den Vektor ***v*** entsteht:

$$
\begin{aligned}
\operatorname{rot}\operatorname{rot}\boldsymbol{v} = {} & \left(\frac{\partial^2 v_y}{\partial x \partial y} - \frac{\partial^2 v_x}{\partial y^2} - \frac{\partial^2 v_x}{\partial z^2} + \frac{\partial^2 v_z}{\partial x \partial z} + \underbrace{\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}}_{A} \right) \boldsymbol{e}_1 \\
& + \left(\frac{\partial^2 v_z}{\partial y \partial z} - \frac{\partial^2 v_y}{\partial z^2} - \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_x}{\partial y \partial x} + \underbrace{\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}}_{B} \right) \boldsymbol{e}_2 \\
& + \left(\frac{\partial^2 v_x}{\partial z \partial x} - \frac{\partial^2 v_z}{\partial x^2} - \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_y}{\partial z \partial y} + \underbrace{\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}}_{C} \right) \boldsymbol{e}_3 \\
= {} & \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_y}{\partial x \partial y} + \frac{\partial^2 v_z}{\partial x \partial z} \right) \boldsymbol{e}_1 + \left(\frac{\partial^2 v_x}{\partial y \partial x} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_z}{\partial y \partial z} \right) \boldsymbol{e}_2 \\
& + \left(\frac{\partial^2 v_x}{\partial z \partial x} + \frac{\partial^2 v_y}{\partial z \partial y} + \frac{\partial^2 v_z}{\partial z^2} \right) \boldsymbol{e}_3 \\
& - \left(\underbrace{\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}}_{A} \right) \boldsymbol{e}_1 - \left(\underbrace{\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}}_{B} \right) \boldsymbol{e}_2 \\
& - \left(\underbrace{\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2}}_{C} \right) \boldsymbol{e}_3 \\
= {} & \operatorname{grad}\operatorname{div}\boldsymbol{v} - (\Delta v_x \boldsymbol{e}_1 + \Delta v_y \boldsymbol{e}_2 + \Delta v_z \boldsymbol{e}_3).
\end{aligned}
$$

Also ist

$$
\operatorname{rot}\operatorname{rot}\boldsymbol{v} = \operatorname{grad}\operatorname{div}\boldsymbol{v} - \Delta\boldsymbol{v} \quad \text{oder}
$$

$$
\nabla \times (\nabla \times \boldsymbol{v}) = \nabla(\nabla \cdot \boldsymbol{v}) - (\nabla \cdot \nabla)\boldsymbol{v}.
$$